

$\mathcal{N} = 2$ MOONSHINE

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ABSTRACT. We construct a model of moonshine phenomenon based on the use of $\mathcal{N} = 2$ superconformal algebra. We consider an extremal Jacobi form of weight 0 and index 2, and expand it in terms of $\mathcal{N} = 2$ massless and massive representations. We find the multiplicities of massive representations are decomposed into a sum of dimensions of irreducible representations of the group $L_2(11)$.

1. Introduction

Study of the elliptic genus in string compactification by use of the superconformal algebras (SCA) was introduced in [13]. In this approach we use the representation theory of SCA, and decompose elliptic genus in terms of characters of SCA. In superconformal algebras there appear BPS (massless) and non-BPS (massive) representations, and massless characters are mock theta functions which possess unusual modular transformation laws [7, 15, 16]. Intrinsic structure of mock theta functions is revealed in [23] (see also [22]).

Recently a phenomenon similar to the famous Monstrous moonshine [5] was discovered in this analysis [12]: it was found that the expansion coefficients of $K3$ elliptic genus in terms of characters of $\mathcal{N} = 4$ SCA are decomposed into a sum of dimensions of irreducible representations of Mathieu group M_{24} . Analogues of McKay–Thompson series in the Monstrous moonshine were constructed [1, 11, 18, 19], and the decompositions into M_{24} representations have been verified up to very high degrees. This phenomenon, sometimes called Mathieu moonshine, combines (mock) modular forms, a sporadic discrete group, and geometry of $K3$ surface in a curious manner. Since M_{24} is the symmetry group of

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Date: September 3, 2012.

an error correcting code (Golay code) [3], such a moonshine phenomenon may be also interesting from the point of view of a possible mechanism of information processing inside black holes.

Quite recently, a generalization of the Mathieu moonshine has been proposed in [2]: authors of [2] consider a sequence of higher dimensional analogues of Mathieu moonshine parametrized by $m = 2, 3, 4, 5, 7$ where $(m-1)|24$: $m = 2$ case corresponds to $K3$ and the original M_{24} moonshine. $m = 3$ corresponds to a 4-dimensional complex manifold. From a general theory of Jacobi forms [17] it is known that the elliptic genera of complex D -dimensional manifold are given by weak Jacobi forms of weight 0 and index $D/2$. When $D = 4$, there exist two independent weak Jacobi forms with weight 0, index 2

$$Z_1(z; \tau) = 48 \left[\left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^4 + \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^4 + \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^4 \right], \quad (1.1)$$

$$Z(z; \tau) = 4 \left[\left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \cdot \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \cdot \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 + \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \cdot \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 \right]. \quad (1.2)$$

A suitable linear combination of these Jacobi forms, $Z_1 + n Z$, will reproduce an elliptic genus of some 4-dimensional complex manifold ($n = 15$ gives the elliptic genus of Hilbert scheme of two points on $K3$ surface $K3^{[2]}$). Coefficient of Z_1 is fixed since it contains the identity representation in the NS sector.

Somewhat surprisingly authors of [2] chose to drop Z_1 and studied Z in isolation. Decomposition of Z in terms of $\mathcal{N} = 4$ characters was known in the literature [8], and it was possible to guess a new moonshine phenomenon which is based on the Mathieu group M_{12} . At higher values of m they apply a similar construction. Drop the analogue of Z_1 and consider a linear combination of other Jacobi forms which possesses a polar term only in the massive representations of the smallest isospin (an extremal Jacobi form) [2]. Thus in these examples we seem to lose the connection to geometry and elliptic genus, however, there still appear interesting new examples of moonshine phenomena (Z may still describe an elliptic genus of a non-compact manifold [14]).

Ordinarily, $\mathcal{N} = 4$ (resp. $\mathcal{N} = 2$) SCA describes the geometry of hyper-Kähler (resp. Calabi–Yau, CY for short) manifolds. When one drops Z_1 , however, it is not quite clear whether $\mathcal{N} = 4$ or $\mathcal{N} = 2$ is the relevant symmetry of the theory. In this article we take up the above example Z at $D = 4$, and decompose it in terms of $\mathcal{N} = 2$ SCA characters [10] instead of $\mathcal{N} = 4$ [9]. This is to see if it is possible to obtain further examples of moonshine phenomena. We in fact find a moonshine phenomenon with respect to the group $L_2(11)$ which is closely related to M_{12} .

2. $\mathcal{N} = 2$ Superconformal Algebras and Character Decomposition

First let us recall the data of representation theory of $\mathcal{N} = 2$ algebra. Representations of the extended $\mathcal{N} = 2$ algebra with central charge $c = 3D$ were studied in [20, 21]. The

characters of the extended algebra are obtained by summing over the spectral flow of irreducible $\mathcal{N} = 2$ characters.

There exist BPS (massless) and non-BPS (massive) representations in the theory, parametrized by the conformal weight h and $U(1)$ charge Q . In the Ramond sector \tilde{R} (with $(-1)^F$ insertion) characters are given as follows.

- massive (non-BPS) representations:

$$h > \frac{D}{8}; Q = \frac{D}{2}, \frac{D}{2} - 1, \dots, -(\frac{D}{2} - 1), -\frac{D}{2} \text{ and } Q \neq 0 (D = \text{even}),$$

$$\begin{aligned} \text{ch}_{D,h,Q>0}^{\tilde{R},\mathcal{N}=2}(z; \tau) &= (-1)^{Q+\frac{D}{2}-1} q^{h-\frac{D}{8}} \frac{i\theta_{11}(z; \tau)}{[\eta(\tau)]^3} e^{2\pi i(Q-\frac{1}{2})z} \\ &\quad \times \sum_{n \in \mathbb{Z}} q^{\frac{D-1}{2}n^2 + (Q-\frac{1}{2})n} (-e^{2\pi iz})^{(D-1)n}, \end{aligned} \quad (2.1)$$

- massless (BPS) representations:

$$h = \frac{D}{8}; Q = \frac{D}{2} - 1, \frac{D}{2} - 2, \dots, -(\frac{D}{2} - 1),$$

$$\begin{aligned} \text{ch}_{D,h=\frac{D}{8},Q\geq 0}^{\tilde{R},\mathcal{N}=2}(z; \tau) &= (-1)^{Q+\frac{D}{2}} \frac{i\theta_{11}(z; \tau)}{[\eta(\tau)]^3} e^{2\pi i(Q+\frac{1}{2})z} \\ &\quad \times \sum_{n \in \mathbb{Z}} q^{\frac{D-1}{2}n^2 + (Q+\frac{1}{2})n} \frac{(-e^{2\pi iz})^{(D-1)n}}{1 - e^{2\pi iz} q^n}, \end{aligned} \quad (2.2)$$

$$\text{and for } h = \frac{D}{8}; Q = \frac{D}{2}$$

$$\begin{aligned} \text{ch}_{D,h=\frac{D}{8},Q=\frac{D}{2}}^{\tilde{R},\mathcal{N}=2}(z; \tau) &= (-1)^D \frac{i\theta_{11}(z; \tau)}{[\eta(\tau)]^3} e^{2\pi i\frac{D+1}{2}z} \\ &\quad \times \sum_{n \in \mathbb{Z}} q^{\frac{D-1}{2}n^2 + \frac{D+1}{2}n} \frac{(1-q) (-e^{2\pi iz})^{(D-1)n}}{(1 - e^{2\pi iz} q^n) (1 - e^{2\pi iz} q^{n+1})}. \end{aligned} \quad (2.3)$$

The characters for $Q < 0$ are given by

$$\text{ch}_{D,h,-Q<0}^{\tilde{R},\mathcal{N}=2}(z; \tau) = \text{ch}_{D,h,Q}^{\tilde{R},\mathcal{N}=2}(-z; \tau). \quad (2.4)$$

The Witten index of massless representations are given by

$$\text{ch}_{D,h=\frac{D}{8},Q\geq 0}^{\tilde{R},\mathcal{N}=2}(z=0; \tau) = \begin{cases} (-1)^{Q+\frac{D}{2}}, & \text{for } 0 \leq Q < \frac{D}{2}, \\ 1 + (-1)^D, & \text{for } Q = \frac{D}{2}, \end{cases} \quad (2.5)$$

while all massive representations having a vanishing index.

At the unitarity bound $h = \frac{D}{8}$, a massive character decomposes into a sum of massless characters as

$$\lim_{h \searrow \frac{D}{8}} \text{ch}_{D,h,Q+1}^{\tilde{R},\mathcal{N}=2}(z; \tau) = \text{ch}_{D,h=\frac{D}{8},Q+1}^{\tilde{R},\mathcal{N}=2}(z; \tau) + \text{ch}_{D,h=\frac{D}{8},Q}^{\tilde{R},\mathcal{N}=2}(z; \tau), \quad (2.6)$$

where $Q \geq 0$, and

$$\begin{aligned} \lim_{h \searrow \frac{D}{8}} \text{ch}_{D,h,Q=\frac{D}{2}}^{\tilde{R},\mathcal{N}=2}(z; \tau) \\ = \text{ch}_{D,h=\frac{D}{8},Q=\frac{D}{2}}^{\tilde{R},\mathcal{N}=2}(z; \tau) + \text{ch}_{D,h=\frac{D}{8},Q=\frac{D}{2}-1}^{\tilde{R},\mathcal{N}=2}(z; \tau) + \text{ch}_{D,h=\frac{D}{8},Q=-(\frac{D}{2}-1)}^{\tilde{R},\mathcal{N}=2}(z; \tau). \end{aligned} \quad (2.7)$$

In our previous paper [10] we pointed out that when the dimension D of CY manifold is odd, the decomposition of its elliptic genus into $\mathcal{N} = 2$ characters becomes essentially the same as the decomposition of the elliptic genus for a corresponding $(D - 3)$ -dimensional hyper-Kähler manifold into $\mathcal{N} = 4$ characters. This is due to the uniqueness of Jacobi form of index $3/2$ and weight 0.

In the case of even D , however, the decomposition of CY manifolds becomes somewhat different from that of hyper-Kähler manifolds. For convenience we introduce functions $B_{D,Q}^{\mathcal{N}=2}(z; \tau)$ and $C_D^{\mathcal{N}=2}(z; \tau)$ by

$$B_{D,Q}^{\mathcal{N}=2}(z; \tau) = (-1)^{Q+\frac{D}{2}-1} q^{-h+\frac{D}{8}+\frac{(Q-\frac{1}{2})^2}{2(D-1)}} \left(\text{ch}_{D,h>\frac{D}{8},Q}^{\tilde{R},\mathcal{N}=2}(z; \tau) + \text{ch}_{D,h>\frac{D}{8},-Q}^{\tilde{R},\mathcal{N}=2}(z; \tau) \right) \\ = \begin{cases} \frac{i\theta_{11}(z; \tau)}{[\eta(\tau)]^3} \sum_{a=Q,D-Q} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2(D-1)}((D-1)n+a-\frac{1}{2})^2} e^{2\pi i((D-1)n+a-\frac{1}{2})z}, \\ \text{for } 1 \leq Q < \frac{D}{2}, \\ \frac{i\theta_{11}(z; \tau)}{[\eta(\tau)]^3} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{D-1}{2}(n+\frac{1}{2})^2} e^{2\pi i(D-1)(n+\frac{1}{2})z}, \\ \text{for } Q = \frac{D}{2}, \end{cases} \quad (2.8)$$

$$C_D^{\mathcal{N}=2}(z; \tau) = (-1)^{\frac{D}{2}} \text{ch}_{D,h=\frac{D}{8},Q=0}^{\tilde{R},\mathcal{N}=2}(z; \tau) \\ = \frac{i\theta_{11}(z; \tau)}{[\eta(\tau)]^3} e^{\pi i z} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{D-1}{2}n^2+\frac{1}{2}n} \frac{e^{2\pi i(D-1)nz}}{1 - e^{2\pi i z} q^n}. \quad (2.9)$$

$B_{D,Q}^{\mathcal{N}=2}$ stands for a charge Q massive character symmetrized under $z \leftrightarrow -z$. $C_D^{\mathcal{N}=2}$ is the massless character for the charge $Q = 0$. The elliptic genus $Z_{CY_D}(z; \tau)$ for the Calabi–Yau D -fold (or any weak Jacobi form of index $D/2$ and weight 0) is decomposed as [10]

$$Z_{CY_D}(z; \tau) = \chi C_D^{\mathcal{N}=2}(z; \tau) + \sum_{a=1}^{D/2} \Sigma_{D,a}(\tau) B_{D,a}^{\mathcal{N}=2}(z; \tau). \quad (2.10)$$

Here χ denotes the Euler number. From a mathematical point of view $\mathcal{N} = 2$ decomposition (2.10) gives a theta series expansion of (a real analytic) Jacobi form with a half-odd integral index [10], while $\mathcal{N} = 4$ decomposition is that of a Jacobi form of an integral index. See also [6] for recent studies of Jacobi forms. Since the massless character $C_D(z; \tau)$ is a mock theta function, the generating functions $\Sigma_{D,a}(\tau)$ for the multiplicity of massive representations become also mock theta functions as far as $\chi \neq 0$.

In the case of $D = 2$, the Calabi–Yau 2-fold is the $K3$ surface. The character decomposition above reduces to the Mathieu moonshine considered in [12]

3. Moonshine from $\mathcal{N} = 2$

Let us now turn to the case $D = 4$, and study a Jacobi form with weight 0 and index 2,

$$\begin{aligned} Z(z; \tau) &= 4 \left[\left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \cdot \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \cdot \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 + \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \cdot \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 \right] \\ &= \frac{1}{12} [\phi_{0,1}(z; \tau)]^2 - \frac{1}{12} E_4(\tau) [\phi_{-2,1}(z; \tau)]^2. \end{aligned} \quad (3.1)$$

As computed in [10], we have a decomposition

$$\begin{aligned} Z(z; \tau) &= 12 C_4^{\mathcal{N}=2}(z; \tau) + \\ &+ q^{-\frac{1}{24}} (-2 + 10q + 20q^2 + 42q^3 + 62q^4 + 118q^5 + 170q^6 + 270q^7 + \dots) B_{4,1}^{\mathcal{N}=2}(z; \tau) \\ &+ q^{-\frac{3}{8}} (12q + 36q^2 + 60q^3 + 120q^4 + 180q^5 + 312q^6 + 456q^7 + \dots) B_{4,2}^{\mathcal{N}=2}(z; \tau) \end{aligned} \quad (3.2)$$

$$\begin{aligned} &= 8 \operatorname{ch}_{D=4, h=\frac{1}{2}, Q=0}^{\tilde{R}, \mathcal{N}=2}(z; \tau) - 2 \operatorname{ch}_{D=4, h=\frac{1}{2}, Q=1}^{\tilde{R}, \mathcal{N}=2}(z; \tau) - 2 \operatorname{ch}_{D=4, h=\frac{1}{2}, Q=-1}^{\tilde{R}, \mathcal{N}=2}(z; \tau) \\ &- \sum_{Q=\pm 1, \pm 2} (-1)^Q \sum_{n=1}^{\infty} p_{|Q|}(n) \operatorname{ch}_{D=4, h=n+\frac{1}{2}, Q}^{\tilde{R}, \mathcal{N}=2}(z; \tau). \end{aligned} \quad (3.3)$$

Expansion coefficients of the massive representations in (3.2) suggest the group $L_2(11)$ being relevant for a moonshine phenomenon. $L_2(11)$ is the group $PSL_2(\mathbb{F}_{11})$ of 2×2 matrices of determinant one with matrix elements in the field \mathbb{F}_{11} [3, 4].

See Table 1 for a character table of $SL_2(11) \cong 2.L_2(11)$, which is a double cover of $L_2(11)$ [4]. Therein n_g denotes the number of elements in conjugacy class g , and the orthogonality relation reads as

$$\sum_g n_g \chi_R^g \overline{\chi_{R'}}^g = |G| \delta_{R, R'}. \quad (3.4)$$

$|G|$ denotes the order of G , and $|SL_2(11)| = 2^3 \cdot 3 \cdot 5 \cdot 11 = 1320$. It is easy to check that at small values of n , the number of massive representations $p_a(n)$ can be written as a sum of dimensions of irreducible representations R of $SL_2(11)$

$$\sum_R \operatorname{mult}_{R,a}(n) \dim R = p_a(n), \quad (3.5)$$

with multiplicities $\operatorname{mult}_{R,a}(n)$

$$\begin{aligned} 10 &= 5 + 5, \quad 20 = 2 \times 10, \quad 42 = 10 + 10 + 2 \times 11, \dots \\ 12 &= 6 + 6, \quad 36 = 6 + 6 + 12 + 12, \quad 60 = 3 \times (6 + 6) + 12 + 12, \dots \end{aligned}$$

It is known that $L_2(11)$ has a permutation representation on 12 symbols (see, *e.g.*, [3]). Representatives of conjugacy classes g are given in Table 2.

As in the case of Mathieu moonshine, we want to construct twisted elliptic genus Z_g for each conjugacy class g . It turns out that due to complication of double covering of the group $L_2(11)$ we can not construct twisted elliptic genera for all classes. However, in the following we obtain those twisted elliptic genera which are just enough to determine the

n_g	1	1	110	132	132	132	132	60	60	110	110	60	60	110	110
$R \setminus g$	1A	2A	4A	5A	5B	10A	10B	11A	11B	12A	12B	22A	22B	3A	6A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	5	5	1	0	0	0	0	$\frac{-1+i\sqrt{11}}{2}$	$\frac{-1-i\sqrt{11}}{2}$	1	1	$\frac{-1-i\sqrt{11}}{2}$	$\frac{-1+i\sqrt{11}}{2}$	-1	-1
χ_3	5	5	1	0	0	0	0	$\frac{-1-i\sqrt{11}}{2}$	$\frac{-1+i\sqrt{11}}{2}$	1	1	$\frac{-1+i\sqrt{11}}{2}$	$\frac{-1-i\sqrt{11}}{2}$	-1	-1
χ_4	10	10	-2	0	0	0	0	-1	-1	1	1	-1	-1	1	1
χ_5	10	10	2	0	0	0	0	-1	-1	-1	-1	-1	-1	1	1
χ_6	11	11	-1	1	1	1	1	0	0	-1	-1	0	0	-1	-1
χ_7	12	12	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	1	1	0	0	1	1	0	0
χ_8	12	12	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	1	1	0	0	1	1	0	0
χ_9	6	-6	0	1	1	-1	-1	$\frac{1-i\sqrt{11}}{2}$	$\frac{1+i\sqrt{11}}{2}$	0	0	$\frac{-1-i\sqrt{11}}{2}$	$\frac{-1+i\sqrt{11}}{2}$	0	0
χ_{10}	6	-6	0	1	1	-1	-1	$\frac{1+i\sqrt{11}}{2}$	$\frac{1-i\sqrt{11}}{2}$	0	0	$\frac{-1+i\sqrt{11}}{2}$	$\frac{-1-i\sqrt{11}}{2}$	0	0
χ_{11}	10	-10	0	0	0	0	0	-1	-1	0	0	1	1	-2	2
χ_{12}	10	-10	0	0	0	0	0	-1	-1	$-\sqrt{3}$	$\sqrt{3}$	1	1	1	-1
χ_{13}	10	-10	0	0	0	0	0	-1	-1	$\sqrt{3}$	$-\sqrt{3}$	1	1	1	-1
χ_{14}	12	-12	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	1	1	0	0	-1	-1	0	0
χ_{15}	12	-12	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	1	1	0	0	-1	-1	0	0

TABLE 1. Character table for $SL_2(11) \cong 2.L_2(11)$ [4].

g	cycle shape	permutation
1A	1^{12}	()
5A	$1^2 5^2$	(3, 5, 7, 9, 11)(4, 6, 8, 10, 12)
5B	$1^2 5^2$	(3, 7, 11, 5, 9)(4, 8, 12, 6, 10)
11A	$1^1 11^1$	(2, 3, 4, 11, 5, 7, 12, 10, 6, 9, 8)
11B	$1^1 11^1$	(2, 4, 5, 12, 6, 8, 3, 11, 7, 10, 9)
4A	2^6	(1, 2)(3, 4)(5, 12)(6, 11)(7, 10)(8, 9)
3A	3^4	(1, 2, 3)(4, 8, 12)(5, 10, 9)(6, 11, 7)
12AB	6^2	(1, 2, 3, 10, 4, 11)(5, 6, 12, 8, 9, 7)

TABLE 2. Permutation representatives of conjugacy classes of $L_2(11)$.
Note that the name of conjugacy class is for that of $SL_2(11)$.

decomposition of the multiplicities of massive representations into the sum of irreducible representations of $SL_2(11)$.

Let us call the representations $\{\chi_i\}$, $i = 1, 2, 3, 4, 5, 6, 7, 8$ in Table 1 as even and representations $\{\chi_j\}$, $j = 9, 10, 11, 12, 13, 14, 15$ as odd, respectively. We assume as in [2] that multiplicities of $|Q| = 1$ massive representations are decomposed into a sum of even representations, and that those of $|Q| = 2$ massive representations are decomposed into a sum of odd representations.

Twisted elliptic genus is a Jacobi form with weight 0 and index 2 and has a decomposition analogous to (3.2),

$$Z_g(z; \tau) = \chi_g C_4^{\mathcal{N}=2}(z; \tau) + \Sigma_{g,1}(\tau) B_{4,1}^{\mathcal{N}=2}(z; \tau) + \Sigma_{g,2}(\tau) B_{4,2}^{\mathcal{N}=2}(z; \tau). \quad (3.6)$$

Here χ_g is the Euler number, $\chi_g = Z_g(0; \tau)$, and $\Sigma_{g,a}(\tau)$ are q -series with integral Fourier coefficients

$$\Sigma_{g,a}(\tau) = q^{-\frac{(2a-1)^2}{24}} \sum_{n=0}^{\infty} p_{g,a}(n) q^n. \quad (3.7)$$

Structure of the character table suggests that the conjugacy classes 5A and 5B have the same twisted elliptic genus, and we use the notation 5AB. Similarly we assume the same for classes 10A, B, 11A, B and 22A, B, and use the notations 10AB, 11AB, and 22AB, respectively.

In view of the character table and the permutation representatives of conjugacy classes, we suppose that the Euler number for class g is given by

$$\chi_g = \chi_1^g + \chi_6^g. \quad (3.8)$$

We then find

g	1A	5AB	11AB	4A	3A	12AB
χ_g	12	2	1	0	0	0

Thus the classes $\{1A, 5AB, 11AB\}$ belong to type I and $\{4A, 3A, 12AB\}$ belong to type II in the terminology of [11].

The original elliptic genus (3.1) is for the class $g = 1A$. By trial and error we have constructed the twisted elliptic genera $Z_g(z; \tau)$ for classes $g = 5AB, 11AB, 4A, 3A, 12AB$, which are presented in Table 3.

g	$Z_g^{\mathcal{N}=2}(z; \tau)$
1A	$\frac{1}{12} [\phi_{0,1}(z; \tau)]^2 - \frac{1}{12} E_4(\tau) [\phi_{-2,1}(z; \tau)]^2$
5AB	$\begin{aligned} & \frac{1}{72} [\phi_{0,1}(z; \tau)]^2 \\ & + \left(-\frac{5}{576} \phi_2^{(3)}(\tau) + \frac{25}{288} \phi_2^{(5)}(\tau) + \frac{35}{576} \phi_2^{(15)}(\tau) + \frac{5}{16} \eta(\tau) \eta(3\tau) \eta(5\tau) \eta(15\tau) \right) \phi_{0,1}(z; \tau) \phi_{-2,1}(z; \tau) \\ & + \left(-\frac{1}{192} E_4(\tau) + \frac{25}{16} [\eta(\tau) \eta(5\tau)]^4 - \frac{75}{4} [\eta(\tau) \eta(3\tau) \eta(5\tau) \eta(15\tau)]^2 \right. \\ & \quad \left. + \frac{5}{96} [\phi_2^{(3)}(\tau)]^2 - \frac{25}{576} [\phi_2^{(5)}(\tau)]^2 - \frac{5}{32} \phi_2^{(3)}(\tau) \phi_2^{(5)}(\tau) \right. \\ & \quad \left. + \frac{175}{96} \phi_2^{(3)}(\tau) \phi_2^{(15)}(\tau) - \frac{175}{96} \phi_2^{(5)}(\tau) \phi_2^{(15)}(\tau) \right) [\phi_{-2,1}(z; \tau)]^2 \end{aligned}$
11AB	$\begin{aligned} & \frac{1}{144} [\phi_{0,1}(z; \tau)]^2 + \left(\frac{11}{72} \phi_2^{(11)}(z; \tau) + \frac{11}{20} [\eta(\tau) \eta(11\tau)]^2 \right) \phi_{-2,1}(z; \tau) \phi_{0,1}(z; \tau) \\ & + \left(\frac{1}{120} E_4(\tau) - \frac{121}{720} [\phi_2^{(11)}(z; \tau)]^2 + \frac{1089}{100} \phi_2^{(11)}(z; \tau) [\eta(\tau) \eta(11\tau)]^2 - \frac{121}{125} [\eta(\tau) \eta(11\tau)]^4 \right) [\phi_{-2,1}(z; \tau)]^2 \end{aligned}$
4A	$-2 \frac{\eta(\tau) \eta(2\tau)}{\eta(4\tau)} B_{4,1}^{\mathcal{N}=2}(z; \tau)$
12AB	$\left(\frac{\eta(\tau) \eta(2\tau)}{\eta(4\tau)} + 3 \frac{[\eta(3\tau)]^2 \eta(6\tau)}{\eta(\tau) \eta(12\tau)} - 6 \frac{\eta(4\tau) [\eta(6\tau)]^4}{\eta(2\tau) \eta(3\tau) [\eta(12\tau)]^2} \right) B_{4,1}^{\mathcal{N}=2}(z; \tau)$
3A	$-2 \frac{[\eta(2\tau)]^3}{\eta(\tau) \eta(6\tau)} B_{4,1}^{\mathcal{N}=2}(z; \tau)$

TABLE 3. Twisted elliptic genus $Z_g(z; \tau)$.

In the case of conjugacy class 2A we assume

$$\begin{aligned} \Sigma_{1A,1}(\tau) &= \Sigma_{2A,1}(\tau), \\ \Sigma_{1A,2}(\tau) &= -\Sigma_{2A,2}(\tau), \end{aligned} \quad (3.9)$$

corresponding to the sign change in the odd sector of character table (see Table 1). We suppose a similar pairing as above (sign change in the $\Sigma_{g,2}$ part) between 5AB and 10AB, 11AB and 22AB.

In the case of 4A, on the other hand, we set the odd part to vanish

$$\Sigma_{4A,2}(\tau) = 0 \quad (3.10)$$

since the odd elements in the character table all vanish (see Table 1) for class 4A. We also assume that the conjugacy classes, 12AB, 3A, and 6A, have vanishing odd parts.

In the case of the Mathieu moonshine, all the twisted elliptic genera were Jacobi forms on congruence subgroup $\Gamma_0(\text{ord}(g))$ with a possible character. In the present case only the twisted elliptic genera of conjugacy classes 1A, 5AB, 11AB, 4A, 12AB, 3A = 6A are Jacobi forms (level of congruence subgroup is sometimes higher than $\text{ord}(g)$). Due to the sign flip in odd sector (3.9) twisted elliptic genera of the other classes can not be Jacobi forms. If we insist that twisted elliptic genera must be Jacobi forms, twisted elliptic genera do not exist for classes 2A, 10AB, 22AB. This situation is similar to the $\mathcal{N} = 4$ moonshine in [2].

In Table 4, the Fourier coefficients of $\Sigma_{g,a}(\tau)$, *i.e.*, the number of the massive representations are given. We have omitted from Table the odd sector $p_{g,2}$ for classes g whose generating functions vanish identically $\Sigma_{g,2}(\tau) = 0$.

In order to test the moonshine conjecture, we have computed multiplicities $\text{mult}_{R,a}(n)$ of representations R

$$p_{g,a}(n) = \sum_R \text{mult}_{R,a}(n) \chi_R^g. \quad (3.11)$$

Here R runs over irreducible representations from χ_1 to χ_8 (resp. from χ_9 to χ_{15}) for $a = 1$ (resp. $a = 2$). From the orthogonality relation (3.4), we have

$$\text{mult}_{R,a}(n) = \sum_g \frac{n_g}{|G|} \overline{\chi_R^g} p_{g,a}(n). \quad (3.12)$$

See Table 5 for the results of the decomposition into irreducible representations. We find that the multiplicities $\text{mult}_{R,1}(n)$ are the same for $R = \chi_2$ and $R = \chi_3$, and also for $R = \chi_7$ and $R = \chi_8$ in the even sector. In the odd sector we have $\chi_9 = \chi_{10}$, $\chi_{11} = \chi_{12} = \chi_{13}$, and $\chi_{14} = \chi_{15}$.

We have verified up to $n = 100$ the positivity and integrality of the multiplicities $\text{mult}_{R,a}(n)$, and consider this to be a strong evidence for a $\mathcal{N} = 2$ moonshine.

4. Discussions

In this paper we have taken up the suggestion of [2] on the extremal Jacobi form. We studied the decomposition of an extremal form of index 2 into characters of $\mathcal{N} = 2$ SCA. We have found a strong evidence for a moonshine phenomenon with respect to the group $L_2(11)$, which is a subgroup of M_{12} which appeared in [2] from decomposition into $\mathcal{N} = 4$ SCA characters.

$n \backslash g$	$p_{g,1}(n)$						$p_{g,2}(n)$		
	1A	5AB	11AB	4A	12AB	3A	1A	5AB	11AB
0	-2	-2	-2	-2	-2	-2	0	0	0
1	10	0	-1	2	2	-2	12	2	1
2	20	0	-2	4	-2	2	36	1	3
3	42	2	-2	-2	-2	0	60	5	5
4	62	2	-4	-2	4	2	120	10	-1
5	118	-2	-3	-2	4	4	180	15	4
6	170	0	-6	2	8	-4	312	27	4
7	270	0	-5	-2	4	0	456	36	5
8	400	0	-7	-4	14	4	720	60	5
9	600	0	-5	4	10	0	1020	85	8
10	828	-2	-8	4	10	0	1524	129	6
11	1220	0	-12	0	12	2	2124	179	12
12	1670	0	-13	-2	22	-4	3036	251	11
13	2330	0	-13	2	14	-4	4140	345	15
14	3162	2	-17	2	20	6	5760	480	18
15	4316	-4	-18	0	30	-4	7740	645	18
16	5730	0	-23	-6	42	0	10512	877	18
17	7710	0	-23	2	38	6	13896	1156	25
18	10102	2	-29	6	42	-8	18540	1545	27
19	13312	2	-31	-4	50	-2	24240	2020	29
20	17298	-2	-38	-6	66	6	31824	2654	34
21	22500	0	-39	0	72	-6	41124	3429	39
22	28860	0	-48	4	70	0	53292	4437	41
23	37162	2	-51	-2	82	10	68220	5685	53
24	47262	2	-60	-6	96	-12	87420	7285	58
25	60128	-2	-64	4	112	-4	110880	9240	66
26	75900	0	-77	8	116	12	140724	11729	67
27	95740	0	-81	-4	128	-8	177072	14752	82
28	119860	0	-95	-4	152	-2	222780	18565	85
29	150062	2	-99	2	170	14	278280	23190	101
30	186576	-4	-116	8	182	-12	347424	28954	110
31	231800	0	-124	-4	206	-4	431136	35931	123
32	286530	0	-141	-10	236	18	534492	44537	134
33	353694	4	-154	6	252	-12	659220	54935	155
34	434524	4	-174	12	270	-2	812160	67680	162
35	533334	-6	-188	-2	310	18	996084	83009	188
36	651790	0	-213	-10	350	-20	1220124	101679	202
37	795490	0	-228	2	380	-8	1488612	124047	224
38	967490	0	-257	10	400	26	1813860	151155	246
39	1174962	2	-278	-6	450	-18	2202420	183535	275
40	1422264	-6	-311	-12	504	0	2670564	222549	292
41	1719450	0	-334	6	546	30	3228048	269008	329
42	2072480	0	-371	12	588	-28	3896568	324708	357
43	2494542	2	-401	-6	648	-6	4690320	390860	393
44	2994874	4	-448	-14	718	34	5637960	469830	427
45	3590404	-6	-480	4	778	-26	6759744	563314	475
46	4294020	0	-534	12	834	-6	8093748	674483	509
47	5128880	0	-574	-4	908	38	9668448	805698	570
48	6112362	2	-635	-18	1002	-36	11534040	961170	606
49	7274774	4	-681	10	1090	-10	13730220	1144185	669
50	8641024	-6	-752	20	1166	46	16323228	1360273	724

TABLE 4. The number of massive representations, $p_{g,1}(n)$ and $p_{g,2}(n)$

$n \setminus R$	$\text{mult}_{R,1}(n)$						$\text{mult}_{R,2}(n)$		
	χ_1	$\chi_2 = \chi_3$	χ_4	χ_5	χ_6	$\chi_7 = \chi_8$	$\chi_9 = \chi_{10}$	$\chi_{11} = \chi_{12} = \chi_{13}$	$\chi_{14} = \chi_{15}$
0	-2	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	1	0	0
2	0	0	0	2	0	0	1	0	1
3	0	0	1	1	2	0	3	0	1
4	1	1	3	1	1	0	5	2	0
5	0	1	4	2	0	2	8	2	1
6	0	4	4	2	2	2	14	4	1
7	0	3	6	4	4	4	19	6	2
8	2	5	11	5	4	6	31	10	2
9	2	7	11	9	8	10	44	14	3
10	1	9	15	13	11	14	66	22	3
11	2	12	23	19	18	20	92	30	5
12	3	18	31	23	25	28	129	44	7
13	3	22	39	35	37	40	177	60	9
14	7	28	55	49	49	54	246	84	12
15	6	40	73	63	66	76	330	114	15
16	11	52	99	83	89	100	448	156	19
17	15	66	128	116	121	136	591	206	26
18	17	88	163	151	163	178	789	276	33
19	23	112	216	198	215	236	1031	362	42
20	30	144	282	258	276	308	1354	476	54
21	38	187	359	335	364	402	1749	616	69
22	47	235	457	435	469	516	2263	800	89
23	63	298	588	560	605	666	2899	1024	113
24	75	381	742	708	775	848	3714	1314	143
25	97	481	940	904	983	1082	4710	1668	180
26	123	600	1184	1148	1243	1366	5977	2120	225
27	150	755	1486	1442	1576	1726	7518	2668	284
28	189	942	1859	1807	1973	2162	9459	3360	353
29	241	1172	2322	2266	2471	2710	11815	4198	440
30	289	1457	2875	2817	3079	3372	14750	5244	546
31	362	1802	3569	3499	3830	4192	18303	6510	675
32	450	2219	4411	4329	4734	5184	22686	8074	835
33	550	2738	5426	5344	5856	6402	27981	9960	1027
34	674	3354	6658	6572	7198	7868	34470	12276	1260
35	826	4106	8170	8066	8832	9664	42276	15058	1543
36	1003	5018	9971	9851	10809	11812	51782	18450	1885
37	1226	6112	12156	12030	13196	14422	63172	22514	2297
38	1491	7416	14775	14645	16053	17544	76974	27438	2793
39	1802	9004	17926	17774	19512	21312	93461	33320	3387
40	2179	10886	21692	21520	23619	25804	113324	40410	4099
41	2641	13143	26208	26028	28561	31202	136979	48850	4950
42	3167	15838	31560	31368	34447	37614	165339	58974	5970
43	3814	19043	37977	37759	41470	45282	199019	70994	7178
44	4582	22842	45586	45342	49792	54370	239225	85346	8620
45	5476	27378	54612	54354	59712	65194	286821	102334	10328
46	6548	32720	65294	65020	71428	77976	343419	122540	12355
47	7824	39052	77973	77669	85324	93148	410226	146388	14754
48	9306	46535	92891	92551	101714	111018	489378	174648	17586
49	11081	55358	110526	110166	121067	132144	582555	207912	20925
50	13157	65719	131260	130878	143811	156974	692568	247190	24863

TABLE 5. Multiplicities $\text{mult}_{R,a}(n)$ up to $n = 50$.

Currently, however, the real origin of the moonshine phenomenon is not very well understood and still remains rather mysterious. It seems that we have to construct and study more examples of moonshine phenomena before we figure out the workings behind them. Especially the $\mathcal{N} = 2$ decomposition of models of [2] for higher values of $m = 4, 5, 7$ may be good candidates of moonshine with the group $L_2(7), L_2(5), L_2(3)$, respectively.

Acknowledgments

This work is supported in part by Grant-in-Aid #23340115, #22224001, #24654041, #22540069 from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

Appendix

A. Modular Forms

As usual we set $q = e^{2\pi i\tau}$ where τ is in the upper half-plane. The Dedekind η -function is

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \quad (\text{A.1})$$

The Eisenstein series $E_{2k}(\tau)$ is

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \left(\sum_{1 \leq r|n} r^{2k-1} \right) q^n, \quad (\text{A.2})$$

where B_k is the Bernoulli number

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

We use modular form of weight 2 on $\Gamma_0(M)$

$$\phi_2^{(M)}(\tau) = \frac{24}{M-1} q \frac{\partial}{\partial q} \log \frac{\eta(M\tau)}{\eta(\tau)}. \quad (\text{A.3})$$

The Jacobi theta functions are defined as

$$\begin{aligned} \theta_{11}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i(n+\frac{1}{2})(z+\frac{1}{2})}, \\ \theta_{10}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i(n+\frac{1}{2})z}, \\ \theta_{00}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi in z}, \\ \theta_{01}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi in(z+\frac{1}{2})}. \end{aligned} \quad (\text{A.4})$$

Some of the Jacobi forms are given by use of these q -series as

$$\phi_{-2,1}(z; \tau) = -\frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^6}, \quad (\text{A.5})$$

$$\phi_{0,1}(z; \tau) = 4 \left[\left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 + \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 \right]. \quad (\text{A.6})$$

See [17] for general properties of the Jacobi forms.

B. $\mathcal{N} = 4$ Moonshine

In order to compare the decomposition in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SCA, we reproduce the analysis of the $m = 3$ case in [2] where the M_{12} moonshine is observed. See Table 6 for the character table of $2.M_{12}$, which is a double cover of M_{12} . The order is $|2.M_{12}| = 2^7 \cdot 3^3 \cdot 5 \cdot 11 = 190080$.

Twisted elliptic genus $Z_g^{\mathcal{N}=4}(z; \tau)$ of $\mathcal{N} = 4$ theory are summarized in Table 7. As in the case of $\mathcal{N} = 2$, not all the conjugacy classes have the elliptic genus. Namely, when g and g' is a pair of classes with the same values of characters in the even sector ($\chi_i^g = \chi_i^{g'}$, for $i = 1, \dots, 15$) and opposite values in the odd sector ($\chi_i^g = -\chi_i^{g'}$, for $i = 16, \dots, 26$), only either Z_g or $Z_{g'}$ is a Jacobi form and becomes a twisted genus.

Note that the $\mathcal{N} = 2$ twisted elliptic genus for type-I classes are somewhat similar to those of $\mathcal{N} = 4$, and we have $Z_g^{\mathcal{N}=2}(z; \tau) = Z_g^{\mathcal{N}=4}(z; \tau)$ for $g = 1A, 11AB$, and

$$Z_{5AB}^{\mathcal{N}=2}(z; \tau) = Z_{5A}^{\mathcal{N}=4}(z; \tau) + 5 \frac{[\eta(15\tau)]^3}{\eta(\tau) \eta(5\tau)} B_{4,2}^{\mathcal{N}=2}(z; \tau). \quad (B.1)$$

In $\mathcal{N} = 4$ SCA, twisted elliptic genera in Table 7 are decomposed as [7, 9]

$$Z_g^{\mathcal{N}=4}(z; \tau) = \chi_g \operatorname{ch}_{k=2, h=\frac{2}{4}, \ell=0}^{\tilde{R}, \mathcal{N}=4}(z; \tau) + \Sigma_g^{(1)}(\tau) B_2^{(1), \mathcal{N}=4}(z; \tau) + \Sigma_g^{(2)}(\tau) B_2^{(2), \mathcal{N}=4}(z; \tau), \quad (B.2)$$

where χ_g is the Witten index, $\chi_g = Z_g^{\mathcal{N}=4}(z=0; \tau)$, and is given by $\chi_g = \chi_1^g + \chi_2^g$,

g	1A	2B	3A	5A	6C	8C	11AB	others
χ_g	12	4	3	2	1	2	1	0

$\mathcal{N} = 4$ massless characters and bases of massive characters are respectively given as

$$\operatorname{ch}_{k=2, h=\frac{2}{4}, \ell=0}^{\tilde{R}, \mathcal{N}=4}(z; \tau) = \frac{\theta_{11}(z; \tau)^2}{\eta(\tau)^3} \frac{i}{\theta_{11}(2z; \tau)} \sum_{n \in \mathbb{Z}} q^{3n^2} e^{12\pi i n z} \frac{1 + q^n e^{2\pi i z}}{1 - q^n e^{2\pi i z}}, \quad (B.3)$$

$$B_2^{(a), \mathcal{N}=4}(z; \tau) = \frac{\theta_{11}(z; \tau)^2}{\eta(\tau)^3} \chi_{1, \frac{a-1}{2}}(z; \tau), \quad (B.4)$$

where $\chi_{1,j}$ is an $SU(2)$ spin j affine character at level 1.

The q -series $\Sigma_g^{(a)}(\tau)$ is the generating function of the number of $\mathcal{N} = 4$ massive representations, and we have

$$\Sigma_g^{(a)}(\tau) = q^{-\frac{a^2}{12}} \sum_{n=0}^{\infty} A_g^{(a)}(n) q^n. \quad (B.5)$$

For comparison with our $\mathcal{N} = 2$ moonshine, values of the Fourier coefficients $A_g^{(a)}(n)$ are given in Tables 8. Note that as in the case of $\mathcal{N} = 2$ the sign change of odd part in the character table is reflected in *e.g.* $\Sigma_{2A}^{(2)}(\tau) = -\Sigma_{1A}^{(2)}(\tau)$. It should be remarked that we have $\Sigma_{4C}^{(1)}(\tau) = \Sigma_{2B}^{(1)}(\tau)$.

Multiplicities of massive representations $A_g^{(a)}(n)$ are given by formula like (3.11) with the character table for $2.M_{12}$ in Table 6. Multiplicities of irreducible representations are completely determined by Table 8.

n_g	1	1	792	495	495	1760	1760	2640	2640	5940	5940	9504	9504	15840	15840	15840	11880	11880	11880	11880	9504	9504	8640	8640	8640	8640
$R \backslash g$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	11	11	-1	3	3	2	2	-1	-1	-1	3	1	1	-1	0	0	-1	-1	1	1	-1	-1	0	0	0	0
χ_3	11	11	-1	3	3	2	2	-1	-1	3	-1	1	1	-1	0	0	1	1	-1	-1	-1	-1	0	0	0	0
χ_4	16	16	4	0	0	-2	-2	1	1	0	0	1	1	1	0	0	0	0	0	0	-1	-1	$\frac{-1+i\sqrt{11}}{2}$	$\frac{-1+i\sqrt{11}}{2}$	$\frac{-1-i\sqrt{11}}{2}$	$\frac{-1-i\sqrt{11}}{2}$
χ_5	16	16	4	0	0	-2	-2	1	1	0	0	1	1	1	0	0	0	0	0	0	-1	-1	$\frac{-1-i\sqrt{11}}{2}$	$\frac{-1-i\sqrt{11}}{2}$	$\frac{-1+i\sqrt{11}}{2}$	$\frac{-1+i\sqrt{11}}{2}$
χ_6	45	45	5	-3	-3	0	0	3	3	1	1	0	0	-1	0	0	-1	-1	-1	-1	0	0	1	1	1	1
χ_7	54	54	6	6	6	0	0	0	0	2	2	-1	-1	0	0	0	0	0	0	0	1	1	-1	-1	-1	-1
χ_8	55	55	-5	7	7	1	1	1	1	-1	-1	0	0	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0
χ_9	55	55	-5	-1	-1	1	1	1	1	3	-1	0	0	1	-1	-1	-1	-1	1	1	0	0	0	0	0	0
χ_{10}	55	55	-5	-1	-1	1	1	1	1	-1	3	0	0	1	-1	-1	1	1	-1	-1	0	0	0	0	0	0
χ_{11}	66	66	6	2	2	3	3	0	0	-2	-2	1	1	0	-1	-1	0	0	0	0	1	1	0	0	0	0
χ_{12}	99	99	-1	3	3	0	0	3	3	-1	-1	-1	-1	-1	0	0	1	1	1	1	-1	-1	0	0	0	0
χ_{13}	120	120	0	-8	-8	3	3	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	-1	-1	-1	-1
χ_{14}	144	144	4	0	0	0	0	-3	-3	0	0	-1	-1	1	0	0	0	0	0	0	-1	-1	1	1	1	1
χ_{15}	176	176	-4	0	0	-4	-4	-1	-1	0	0	1	1	-1	0	0	0	0	0	0	1	1	0	0	0	0
χ_{16}	10	-10	0	-2	2	1	-1	-2	2	0	0	0	0	0	1	-1	$i\sqrt{2}$	$-i\sqrt{2}$	$i\sqrt{2}$	$-i\sqrt{2}$	0	0	-1	1	-1	1
χ_{17}	10	-10	0	-2	2	1	-1	-2	2	0	0	0	0	0	1	-1	$-i\sqrt{2}$	$i\sqrt{2}$	$-i\sqrt{2}$	$i\sqrt{2}$	0	0	-1	1	-1	1
χ_{18}	12	-12	0	4	-4	3	-3	0	0	0	0	2	-2	0	1	-1	0	0	0	0	0	0	1	-1	1	-1
χ_{19}	32	-32	0	0	0	-4	4	2	-2	0	0	2	-2	0	0	0	0	0	0	0	0	0	-1	1	-1	1
χ_{20}	44	-44	0	4	-4	-1	1	2	-2	0	0	-1	1	0	1	-1	0	0	0	0	$i\sqrt{5}$	$-i\sqrt{5}$	0	0	0	0
χ_{21}	44	-44	0	4	-4	-1	1	2	-2	0	0	-1	1	0	1	-1	0	0	0	0	$-i\sqrt{5}$	$i\sqrt{5}$	0	0	0	0
χ_{22}	110	-110	0	-6	6	2	-2	2	-2	0	0	0	0	0	0	0	$i\sqrt{2}$	$-i\sqrt{2}$	$-i\sqrt{2}$	$i\sqrt{2}$	0	0	0	0	0	0
χ_{23}	110	-110	0	-6	6	2	-2	2	-2	0	0	0	0	0	0	0	$-i\sqrt{2}$	$i\sqrt{2}$	$i\sqrt{2}$	$-i\sqrt{2}$	0	0	0	0	0	0
χ_{24}	120	-120	0	8	-8	3	-3	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	-1	1	-1	1
χ_{25}	160	-160	0	0	0	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1-i\sqrt{11}}{2}$	$\frac{-1+i\sqrt{11}}{2}$	$\frac{1+i\sqrt{11}}{2}$	$\frac{-1-i\sqrt{11}}{2}$
χ_{26}	160	-160	0	0	0	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1+i\sqrt{11}}{2}$	$\frac{-1-i\sqrt{11}}{2}$	$\frac{1-i\sqrt{11}}{2}$	$\frac{-1+i\sqrt{11}}{2}$

TABLE 6. Character table of $2.M_{12}$ [4].

g	$Z_g^{\mathcal{N}=4}(z; \tau)$
1A	$\frac{1}{12} [\phi_{0,1}(z; \tau)]^2 - \frac{1}{12} E_4(\tau) [\phi_{-2,1}(z; \tau)]^2$
2B	$\frac{1}{36} [\phi_{0,1}(z; \tau)]^2 + \frac{1}{9} \phi_2^{(2)}(\tau) \phi_{0,1}(z; \tau) \phi_{-2,1}(z; \tau) + \left(-\frac{5}{36} E_4(\tau) + \frac{128}{3} \frac{[\eta(2\tau)]^{16}}{[\eta(\tau)]^8} \right) [\phi_{-2,1}(z; \tau)]^2$
3A	$\frac{1}{48} [\phi_{0,1}(z; \tau)]^2 + \frac{1}{8} \phi_2^{(3)}(\tau) \phi_{0,1}(z; \tau) \phi_{-2,1}(z; \tau) + \left(\frac{11}{48} [\phi_2^{(3)}(\tau)]^2 - \frac{3}{8} E_4(3\tau) \right) [\phi_{-2,1}(z; \tau)]^2$
5A	$\frac{1}{72} [\phi_{0,1}(z; \tau)]^2 + \frac{5}{36} \phi_2^{(5)}(\tau) \phi_{0,1}(z; \tau) \phi_{-2,1}(z; \tau) + \left(\frac{1}{48} E_4(\tau) - \frac{25}{144} [\phi_2^{(5)}(\tau)]^2 + \frac{25}{4} [\eta(\tau)\eta(5\tau)]^4 \right) [\phi_{-2,1}(z; \tau)]^2$
6C	$\frac{1}{144} [\phi_{0,1}(z; \tau)]^2 + \left(\frac{5}{24} \phi_2^{(6)}(\tau) - \frac{1}{24} \phi_2^{(3)}(\tau) - \frac{1}{72} \phi_2^{(2)}(\tau) \right) \phi_{0,1}(z; \tau) \phi_{-2,1}(z; \tau) + \left(\frac{1}{36} [\phi_2^{(2)}(\tau)]^2 + \frac{1}{16} [\phi_2^{(3)}(\tau)]^2 + \frac{5}{4} \phi_2^{(2)}(\tau) \phi_2^{(6)}(\tau) - \frac{5}{4} \phi_2^{(3)}(\tau) \phi_2^{(6)}(\tau) - \frac{1}{4} \phi_2^{(2)}(\tau) \phi_2^{(3)}(\tau) \right) [\phi_{-2,1}(z; \tau)]^2$
8C	$\frac{1}{72} [\phi_{0,1}(z; \tau)]^2 + \left(\frac{5}{72} \phi_2^{(2)}(\tau) - \frac{1}{8} \phi_2^{(4)}(\tau) + \frac{7}{36} \phi_2^{(8)}(\tau) \right) \phi_{0,1}(z; \tau) \phi_{-2,1}(z; \tau) + \left(\frac{1}{4} [\phi_2^{(4)}(\tau)]^2 - \frac{5}{24} \phi_2^{(2)}(\tau) \phi_2^{(4)}(\tau) + \frac{35}{36} \phi_2^{(2)}(\tau) \phi_2^{(8)}(\tau) - \frac{7}{6} \phi_2^{(4)}(\tau) \phi_2^{(8)}(\tau) \right) [\phi_{-2,1}(z; \tau)]^2$
11AB	$\frac{1}{144} [\phi_{0,1}(z; \tau)]^2 + \left(\frac{11}{72} \phi_2^{(11)}(z; \tau) + \frac{11}{20} [\eta(\tau)\eta(11\tau)]^2 \right) \phi_{-2,1}(z; \tau) \phi_{0,1}(z; \tau) + \left(\frac{1}{120} E_4(\tau) - \frac{121}{720} [\phi_2^{(11)}(z; \tau)]^2 + \frac{1089}{100} \phi_2^{(11)}(z; \tau) [\eta(\tau)\eta(11\tau)]^2 - \frac{121}{125} [\eta(\tau)\eta(11\tau)]^4 \right) [\phi_{-2,1}(z; \tau)]^2$
4A	$-2 \frac{[\eta(\tau)]^4}{[\eta(2\tau)]^3} B_2^{(1), \mathcal{N}=4}(z; \tau)$
3B	$\left(\frac{1}{6} \phi_2^{(9)}(\tau) - \frac{9}{2} \frac{[\eta(9\tau)]^6}{[\eta(3\tau)]^2} \right) \phi_{-2,1}(z; \tau) \phi_{0,1}(z; \tau) + \left(\frac{27}{2} [\eta(3\tau)]^8 - \frac{1}{6} \frac{[\eta(\tau)]^{12}}{[\eta(3\tau)]^4} \right) [\phi_{-2,1}(z; \tau)]^2$
4B	$-2 \frac{[\eta(2\tau)]^3}{[\eta(4\tau)]^2} B_2^{(1), \mathcal{N}=4}(z; \tau) = \left(\frac{1}{36} \phi_2^{(2)}(\tau) - \frac{1}{4} \phi_2^{(4)}(\tau) + \frac{7}{18} \phi_2^{(8)}(\tau) \right) \phi_{0,1}(z; \tau) \phi_{-2,1}(z; \tau) + \left(\frac{5}{36} [\phi_2^{(2)}(z; \tau)]^2 + \frac{3}{2} [\phi_2^{(4)}(z; \tau)]^2 - \frac{17}{12} \phi_2^{(2)}(z; \tau) \phi_2^{(4)}(z; \tau) + \frac{35}{18} \phi_2^{(2)}(z; \tau) \phi_2^{(8)}(z; \tau) - \frac{7}{3} \phi_2^{(4)}(z; \tau) \phi_2^{(8)}(z; \tau) \right) [\phi_{-2,1}(z; \tau)]^2$
12A	$-2 \frac{\eta(\tau) \eta(3\tau)}{\eta(6\tau)} B_2^{(1), \mathcal{N}=4}(z; \tau)$
8A	$-2 \frac{[\eta(4\tau)]^4}{\eta(2\tau) [\eta(8\tau)]^2} B_2^{(1), \mathcal{N}=4}(z; \tau)$
20A	$-2 \frac{[\eta(2\tau)]^2 \eta(5\tau)}{\eta(\tau) \eta(10\tau)} B_2^{(1), \mathcal{N}=4}(z; \tau)$

TABLE 7. Twisted elliptic genus for $\mathcal{N} = 4$ SCA.

TABLE 8. The number of massive representations $A_g^{(a)}(n)$.

$n \backslash g$	$A_g^{(1)}(n)$													$A_g^{(2)}(n)$							
	1A	2B	3A	5A	6C	8C	11AB	4A	3B	4B	12A	8A	20A	1A	2B	3A	5A	6C	8C	11AB	3B
0	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	0	0	0	0	0	0	0	0
1	32	0	-4	2	0	0	-1	8	2	0	2	0	-2	20	-4	2	0	2	-2	-2	-4
2	110	-2	2	0	-2	2	0	-10	2	6	2	-2	0	88	8	-2	-2	2	4	0	4
3	288	0	0	-2	0	0	2	8	-6	0	2	0	-2	220	-12	4	0	0	-6	0	4
4	660	4	-6	0	-2	0	0	-20	6	-4	-2	4	0	560	16	2	0	-2	8	-1	-4
5	1408	0	4	-2	0	0	0	32	4	0	-4	0	2	1144	-24	-8	4	0	-12	0	4
6	2794	-6	4	4	0	-2	0	-30	-8	2	0	2	0	2400	32	6	0	2	16	2	0
7	5280	0	-12	0	0	0	0	40	6	0	-2	0	0	4488	-40	6	-2	2	-20	0	-12
8	9638	6	8	-2	0	-2	2	-58	2	-10	2	-2	2	8360	56	-10	0	2	28	0	8
9	16960	0	4	0	0	0	-2	80	-14	0	2	0	0	14696	-72	8	-4	0	-36	0	8
10	29018	-6	-16	-2	0	2	0	-102	8	10	0	2	-2	25544	88	2	4	-2	44	2	-16
11	48576	0	12	6	0	0	0	112	6	0	-2	0	2	42660	-116	-18	0	-2	-58	2	12
12	79530	10	6	0	-2	2	0	-150	-24	-6	0	2	0	70576	144	16	-4	0	72	0	4
13	127776	0	-24	-4	0	0	0	200	18	0	2	0	0	113520	-176	12	0	4	-88	0	-24
14	202050	-14	18	0	-2	-2	2	-230	12	10	4	2	0	180640	224	-26	0	2	112	-2	16
15	314688	0	12	-2	0	0	0	272	-30	0	2	0	2	281808	-272	18	8	-2	-136	-1	12
16	483516	12	-36	6	0	0	0	-348	24	-12	0	-4	2	435160	328	10	0	-2	164	0	-32
17	733920	0	24	0	0	0	0	440	12	0	-4	0	0	661476	-404	-42	-4	-2	-202	2	24
18	1101364	-12	16	-6	0	4	0	-508	-44	20	-4	-4	2	996600	488	30	0	2	244	0	12

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